MATH 3060 Assignment 6 solution

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1. Let $x_n = 0, y_n = \frac{\pi}{2n}$, then $|x_n - y_n| \to 0$ but $\cos(nx_n) - \cos(ny_n) = 1$.

2. Let $\epsilon > 0$, choose $\delta > 0$ s.t. $|\phi(x) - \phi(y)| < \epsilon$ whenever $|x - y| < \delta$. Then clearly $|f_n(x) - f_n(y)| < \epsilon$ when $|x - y| < \delta$.
   If ϕ is not uniformly continuous, then $f_n = \phi(x - n)$ is also not uniformly continuous, so $\{f_n\}$ is not equicontinuous.

3. Let $\{f_{n,k,1}\}$ be a subsequence of f_n that converges uniformly on $[-1, 1]$, and let $\{f_{n,k,1}\}$ be a subsequence of $\{f_{n,k,1}\}$ that converges uniformly on $[-2, 2]$ and so on. Now define $f_{n,k} = f_{n,k,k}$, then $\{f_{n,k}\}$ converges pointwise on $\mathbb{R}$.

4. $F(x)| = |\int_0^x |f_n|| \leq |\int_0^1 |f_n|| \leq \sqrt{M}$ by Cauchy Schwartz inequality. Now we want to show $\{F_n\}$ is equicontinuous. Let $x < y \in [0, 1]$, then

$$\begin{aligned} |F(x) - F(y)| &= |\int_y^x f_n| \\ &= |\int_y^x (f_n \cdot 1)| \\ &= \sqrt{\int_y^x f_n^2 \int_y^x 1^2} \\ &\leq \sqrt{M}|x - y|^{1/2}, \end{aligned}$$

   so $\{F_n\}$ is in fact Hölder continuous with the constants independent of n.